Extra Notes 1

Math 652

February 15, 2016

Proposition. *With the notation from class [on 02/12]:*

$$K[\alpha]e_i(\alpha) \cong K[X]/f_i(X) \cdot K[X].$$

Proof. Let $\phi : K[X] \to K[\alpha_i]e(\alpha_i)$ defined by $\phi(g) \stackrel{\text{def}}{=} g(\alpha)e_i(\alpha)$. Since $e_i(\alpha)^2 = e_i(\alpha)$ [idempotent], this map is a homomorphism of rings. [Note also that $1_{K[\alpha]e_i(\alpha)} = e_i(\alpha)$.] It's also clearly onto. So, suffices to show that $\ker\phi = f_i(X) \cdot K[X]$.
Remember:

- $g_i \stackrel{\text{def}}{=} \prod_{j \neq i} f_j$;
- since $(g_1, \ldots, g_n) = 1$ and so there are $h_i \in K[X]$ such that $1 = \sum_i g_i h_i$;
- $e_i \stackrel{\text{def}}{=} g_i h_i$.

So, $e_i f_i = (f_i g_i)h_i = f h_i$, and hence $\phi(f_i) = 0$ [as $f(\alpha) = 0$], and so $f_i \cdot K[X] \subseteq \ker\phi$. But, f_i is irreducible [by definition] and hence $f_i \cdot K[X]$ is a maximal ideal. If $\ker\phi \neq f_i \cdot K[X]$, then it would be all of $K[X]$, but that is not the case as $\phi(1) = e_i(\alpha) \neq 0$ [as proved in class].

$\square$