March 28th, 2018

Math 351

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Spring 2018

Name: ....

Student ID (last 6 digits): XXX- ....

Midterm 3

You have 50 minutes to complete the exam. Do all work on this exam, i.e., on the page of the respective assignment. Indicate clearly, when you continue your solution on the back of the page or another part of the exam.

Write your name and the last six digits of your student ID number on the top of this page. Check that no pages of your exam are missing. This exam has 4 questions and 7 printed pages (including this one, a page for scratch work, and a page with ring axioms in the end).

No calculators, books or notes are allowed on this exam, but you can use your own index cards!

Show all work! (Unless I say otherwise.) Correct answers without work will receive **zero**. Also, **points will be taken from messy solutions**.

Good luck!

Question	Max. Points	Score
1	20	
2	30	
3	20	
4	30	
Total	100	

1) [20 points] If u is a unit in a *commutative* ring, prove that its inverse is unique: if $ua = 1$ and $ub = 1$, then $a = b$. *Justify every step! (Don't skip steps!)* [The axioms are listed in the last page.]

[**Note:** This was a HW problem.]

2) Let R be a commutative ring and $U(R)$ be the set of units of a ring, i.e.,

$$U(R) = \{a \in R \,:\, \exists b \in R \text{ such that } ab = 1\}.$$

[We denote this b, such that $ab = 1$, by a^{-1}.]
[**Hint:** To show $x \in U(R)$ we need to find $y \in R$ such that $xy = 1$.]

(a) [15 points] Show that if $x \in U(R)$, then $x^{-1} \in U(R)$.

(b) [15 points] Show that if $x, y \in U(R)$, then $xy \in U(R)$.

3) [20 points] Let F be a field and suppose that $\mathbb{F}_3$ is a subfield of F. Prove that the prime field of F is $\mathbb{F}_3$. [I proved this in class (in more generality), but you can't use it here, of course. It's a very simple proof though!]

4) Let R be a ring with $\mathbb{F}_2 = \{[0], [1]\}$ [also denoted $\mathbb{I}_2$ or $\mathbb{Z}/2\mathbb{Z}$] as a subring, and having exactly four elements, say $R = \{[0], [1], a, b\}$. [So, no two among $[0]$, $[1]$, a, and b are equal! Hence, $a \neq b$, $a \neq [1]$, $b \neq [0]$, etc.]

(a) [15 points] Prove that since R contains $\mathbb{F}_2$, we have that $2x = 0$ [or $x + x = 0$] for all $x \in R$. [**Hint:** Use the ring axioms [and the fact that $\mathbb{F}_2$ is a subring, of course]. The axioms are given in the last page.]

(b) [15 points] Prove that $a + [1] = b$. [**Hint:** Prove that $a + [1] \neq [0]$, $a + [1] \neq [1]$, and $a + [1] \neq a$.]

Scratch:

Commutative Ring Axioms: A [non-empty] set with two operations, $+$ and $\cdot$, is a commutative ring if:

0. For all $a, b \in R$ we have that $a + b \in R$ and $a \cdot b \in R$.
1. For all $a, b \in R$ we have that $a + b = b + a$.
2. For all $a, b, c \in R$ we have that $(a + b) + c = a + (b + c)$.
3. There exists $0 \in R$ such that for all $a \in R$ we have $a + 0 = a$.
4. For all $a \in R$ there exists $-a \in R$ such that $a + (-a) = 0$.
5. For all $a, b \in R$ we have that $a \cdot b = b \cdot a$.
6. For all $a, b, c \in R$ we have that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
7. There is $1 \in R$ such that for all $a \in R$ we have that $1 \cdot a = a$
8. For all $a, b, c \in R$ we have that $a \cdot (b + c) = a \cdot b + a \cdot c$